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# Stability of a layer of liquid flowing down an inclined plane

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SUMMARY
The stability of the flow down an inclined plane is studied for small angles of inclination $\beta$. The same problem has been studied by S. P. Lin, however using an incorrect boundary condition. The correctly formulated eigenvalue-problem is solved by a numerical integration of the Orr–Sommerfeld equation employing the orthonormalization technique. It is shown that in the range $3' < \beta < 1°$ a decrease in $\beta$ means a decrease in the critical Reynolds number for the hard mode, which is a shear wave modified by the presence of the free surface. In that range the stability is still more or less governed by the stability of the soft waves, which are essentially surface waves modified by the presence of shear.

For values of $\beta < 1'$ the stability is governed by the hard mode, contradictory to Lin's statements. In that case instability occurs at high Reynolds numbers.

## 1. Introduction

The problem of the stability of a layer of liquid flowing down an inclined plane under the action of gravity has been studied by several authors [1], [2], [4–7], [12], [13].

Yih solved the problem, however restricting himself to long waves, the so-called "soft" modes. These waves are essentially surface waves being slightly modified by viscosity. Yih, [12], [13] obtained as the critical Reynolds number, above which instabilities may occur

$$\mathrm{Re}_{cr} = \tfrac{5}{4} \cot\!an\, \beta\,, \tag{1.1}$$

where Re is based on the undisturbed velocity at the free surface and where $\beta$ is the angle of inclination of the plane. The same result has also been obtained by Benjamin [1] in an entirely different way.

Qualitatively the behaviour of the neutral curve is given in figure 1. The neutral curve consists of the entire Re-axis and a curve emanating from the bifurcation point

$$(\alpha, \mathrm{Re}) = (0, \tfrac{5}{4} \,\mathrm{cotan}\, \beta)\,. \tag{1.2}$$

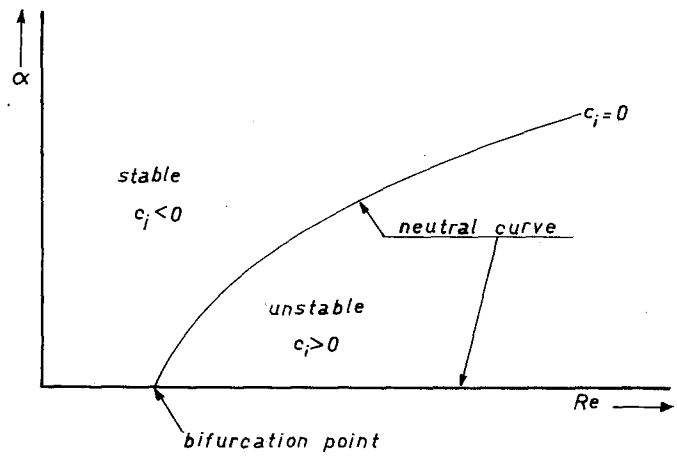


Figure 1. Neutral curve of the soft mode.

Most of the fore-mentioned authors give solutions which are only valid for small wave numbers, i.e. very long waves. However, there are two exceptions. The first is Graef [4], who gives numerical results for wave numbers which are still small, but not extremely small. The second exception is Sung P. Lin, who gives the results for the "hard" mode, which is essentially a Tollmien–Schlichting shear wave modified by the presence of the free surface. In formulating the eigenvalue-problem Lin [6, 7] made a mistake in one of the boundary conditions.

In the present paper the problem is solved, now with correct boundary conditions. The actual method of solution is entirely different from Lin's method, because of the availability of a standard set of subroutines for the solution of the linear stability problem.

## 2. Formulation of the eigen-value problem

In non-dimensional form the steady laminar flow down an inclined plane is given by

$$U(y) = 1 - y^2 . \tag{2.1}$$

The velocity has been made dimensionless with the speed at the free surface

$$U_0 = \frac{d^2 g \sin\beta}{2\nu} , \tag{2.2}$$

whereas length has been made dimensionless with the thickness $d$ of the liquid film, see fig. 2.

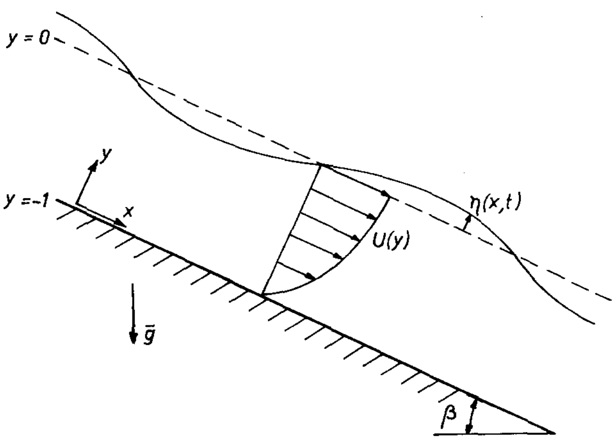


Figure 2. Definition sketch.

The Reynolds number is defined as

$$\mathrm{Re} = \frac{U_0 d}{\nu} = \frac{d^3 g \sin\beta}{2\nu^2} . \tag{2.3}$$

In order to investigate the stability of the steady flow (2.1) against two-dimensional disturbances, one assumes for the disturbance

the streamfunction $\psi(x, y, t) = \phi(y)\,e^{i\alpha(x-ct)}$ , (2.4)

the pressure $p(x, y, t) = \pi(y)\,e^{i\alpha(x-ct)}$ , (2.5)

and the surface elevation $\eta(x, t) = a\,e^{i\alpha(x-ct)}$ . (2.6)

Restriction to two-dimensional disturbances has been made, since Squire's theorem is also valid for fluid motion with a free surface, see [13]. In these expressions $\alpha$ is the wave number and $c$ is the wave velocity. In this paper $\alpha$ is chosen to be real, whereas $c$ is complex, so the temporal stability is investigated, not the spatial. Substituting these equations into the momentum equations, eliminating the pressure and linearizing gives the well-known Orr-Sommerfeld equation for the streamfunction $\phi(y)$

$$(U-c)(\phi''-\alpha^2\phi)-U''\phi = \frac{-i}{\alpha \mathrm{Re}}[\phi^{(4)}-2\alpha^2\phi''+\alpha^4\phi], \tag{2.7}$$

where the prime means differentiation with respect to $y$.

Substituting the steady flow eq. (2.1) gives

$$(1-c-y^2)(\phi''-\alpha^2\phi)+2\phi = \frac{-i}{\alpha \mathrm{Re}}[\phi^{(4)}-2\alpha^2\phi''+\alpha^4\phi]. \tag{2.8}$$

Now the necessary boundary conditions must be derived.

At the bottom of the layer the velocities $u$ and $v$ must be zero, so at $y=-1$

$$v = -\frac{\partial\psi}{\partial x} = 0 \text{ gives } \phi(-1)=0, \tag{2.9}$$

and

$$u = \frac{\partial\psi}{\partial y} = 0 \text{ gives } \phi'(-1)=0, \tag{2.10}$$

where $u$ and $v$ are the disturbance velocities. The remaining boundary conditions are to be fulfilled at the free surface.

The kinematic condition at the free surface requires

$$\frac{D}{Dt}(y-\eta)=0 \text{ at } y=\eta(x,t), \tag{2.11}$$

where $D/Dt$ denotes the material derivative. Working out the last equation gives

$$v-\frac{\partial\eta}{\partial t}-(u+U)\frac{\partial\eta}{\partial x} = 0 \text{ at } y=\eta(x,t). \tag{2.12}$$

Substituted of the assumed disturbance (2.4) through (2.6) and linearization gives

$$a = \frac{\phi(0)}{c-1}. \tag{2.13}$$

This expression gives the phase-shift between the disturbance velocities and the corresponding surface elevation.

At the free surface the stress must vanish neglecting the viscosity of the air above the liquid. In a linear theory the normal and tangential stresses can be replaced by the stresses along the coordinate axes. The condition on the tangential stress is

$$\sigma_{xy}=0 \text{ at } y=\eta(x,t), \tag{2.14}$$

or, equivalently,

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} = 0 \text{ at } y=\eta(x,t). \tag{2.15}$$

Substitution of the primary flow and the disturbance gives after linearization

$$\phi''(0)+\left(\alpha^2-\frac{2}{c-1}\right)\phi(0)=0. \tag{2.16}$$

The replacement of $\sigma_t$ by $\sigma_{xy}$ in eq. (2.14) is allowed since the elevation is much less than the wavelength. Similarly the normal stress $\sigma_n$ can be replaced by $\sigma_{yy}$.

The condition on the normal stress requires

$$\sigma_{yy}=0 \text{ at } y=\eta(x,t), \tag{2.17}$$

the atmospheric pressure being taken equal to zero. The last equation can be expressed in terms of pressure and velocity gradients

$$-P-p+\frac{2}{\mathrm{Re}}\frac{\partial v}{\partial y}=0 \text{ at } y=\eta(x,t), \tag{2.18}$$

where $p$ is the disturbance pressure and $P$ is the steady hydrostatic pressure. Linearization gives

$$-\eta\frac{\mathrm{d}P}{\mathrm{d}y}-p+\frac{2}{\mathrm{Re}}\frac{\partial v}{\partial y}=0 \text{ at } y=0, \tag{2.19}$$

or

$$\frac{2}{\mathrm{Re}}\operatorname{cotan}\beta\,\frac{\phi(0)}{c-1}-\pi(0)-\frac{2\mathrm{i}\alpha}{\mathrm{Re}}\phi'(0)=0. \tag{2.20}$$

Another expression for the disturbance pressure $\pi(0)$ can be found by applying the linearized momentum equation in $x$-direction at $y=0$

$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}+v\frac{\partial U}{\partial y}=-\frac{\partial p}{\partial x}-\frac{\partial P}{\partial x}+\frac{1}{\mathrm{Re}}\Delta u+\frac{1}{\mathrm{Re}}\Delta U+\frac{\sin\beta}{Fr^2} \text{ at } y=0,$$

with $Fr=U_0/(gd)^{\frac{1}{2}}$. Subtracting the steady flow gives (2.21)

$$(1-c)\phi'(0)=-\pi(0)+\frac{1}{\mathrm{i}\alpha\mathrm{Re}}\{\phi'''(0)-\alpha^2\phi'(0)\}. \tag{2.22}$$

Eliminating the pressure $\pi(0)$ one arrives at

$$-\mathrm{i}\phi'''(0)+\alpha[\mathrm{Re}(c-1)+3\mathrm{i}\alpha]\,\phi'(0)-2\alpha\operatorname{cotan}\beta\,\frac{\phi(0)}{c-1}=0. \tag{2.23}$$

S. P. Lin [6, 7] gets the same condition apart from a plus-sign in front of the last term. Apparently this is not a printing error in his papers, but is used consistently in Lin's subsequent calculations using an asymptotic series solution.

Therefore the eigenvalue problem is solved again in this paper, however in an entirely different way employing the orthonormalization technique of Godunov [3]. This technique has been applied by Wazzan *et al.* [11] to stability calculations of boundary layers.

Since the differential equation and the boundary conditions are homogeneous, one is still free to impose a suitable normalization condition on the solution. The condition being used here is the same as used by Thomas [10] for plane Poiseuille flow. The streamfunction $\phi$ is taken equal to unity at the centerline in plane Poiseuille flow, which corresponds to the free surface in this problem. So the eigenvalue problem to be solved is

$$\phi^{(4)}-2\alpha^2\phi''+\alpha^4\phi-\mathrm{i}\alpha\mathrm{Re}[(1-y^2-c)(\phi''-\alpha^2\phi)+2\phi]=0, \tag{2.24a}$$

$$\phi(-1)=0, \tag{2.24b}$$

$$\phi'(-1)=0, \tag{2.24c}$$

$$\phi(0)=1, \tag{2.24d}$$

$$\phi''(0)=\frac{2}{c-1}-\alpha^2, \tag{2.24e}$$

$$-\mathrm{i}\phi'''(0)+\alpha[\mathrm{Re}(c-1)+3\mathrm{i}\alpha]\phi'(0)=\frac{2\alpha\operatorname{cotan}\beta}{c-1}. \tag{2.24f}$$

If the parameters $\alpha$, Re and $\beta$ are kept fixed, the fourth order Orr–Sommerfeld equation (2.24a) plus the five boundary conditions (2.24b) through (2.24f) completely determine the eigenvalue $c$ and the corresponding eigenfunction $\phi$. One can also choose *e.g.* the parameters Re, $\beta$ and $c_i=\mathrm{Im}\{c\}$. In that case the eigenvalue to be looked for is the combination $(c_r, \alpha)$, where $c_r=\mathrm{Re}\{c\}$. If $c_i$ is taken equal to zero, one obtains the neutral curve.

## 3. Numerical solution of the eigenvalue-problem

The eigenvalue-problem has boundary conditions at two points. Since a Runge–Kutta integration method has been chosen, the problem has to be transformed into an initial-value problem in the way to be described next.

Let $\bar{\phi}(y)$ be the four-dimensional complex vector

$$\bar{\phi}(y) = (\phi(y), \phi'(y), \phi''(y), \phi'''(y)) . \tag{3.1}$$

Note the difference between $\bar{\phi}(y)$ and its first component $\phi(y)$.

Define two sets of initial values

$$\bar{\phi}_1(-1) = (0, 0, 1, 0) \tag{3.2}$$

and

$$\bar{\phi}_2(-1) = (0, 0, 0, 1) . \tag{3.3}$$

With an estimated value of $c$ the differential equation (2.24a) can be solved subject to the initial values either given by (3.2) or by (3.3), resulting in the particular solutions $\bar{\phi}_1(y)$ and $\bar{\phi}_2(y)$.

The solution of the system (2.24a) through (2.24f) can be approximated by

$$\bar{\phi}(y) = A_1 \bar{\phi}_1 + A_2 \bar{\phi}_2 . \tag{3.4}$$

By their construction $\bar{\phi}_1(y)$ and $\bar{\phi}_2(y)$ satisfy the boundary conditions at $y = -1$ and so does $\phi(y)$. The normalization condition (2.24d) plus the condition (2.24e) on the tangential stress at the free surface determine the constants $A_1$ and $A_2$.

However, the solution $\phi(y)$ constructed in this way in general does not satisfy the remaining condition (2.24f) on the normal stress. The value of $c$ must be varied in order to satisfy the last condition.

If one calls the left-hand side of equation (2.24f) the complex quantity RESIDU, the zero's of RESIDU are to be sought. Owing to numerical inaccuracies one can't look for zero's of RESIDU, but only for minima, which must be close to zero within certain bounds.

For fixed values of Re, $\alpha$ and $\beta$ a search in the complex $c$-plane is made for minima of RESIDU. To save computing time, the calculation is stopped when RESIDU is equal to zero within some prescribed tolerance and when the subsequent approximations are also within some tolerance.

Tracing the root in the complex $c$-plane has been done by a method which bears a close resemblance to a method of Powell [8] for minimizing a function of several variables without using derivatives (the determination of these is very time consuming). The function to be minimized is |RESIDU|.

Powell's complicated procedure of scanning the complex $c$-plane was found to be necessary both to save computing time and to ensure, that the calculation is not stopped too early. It is still more compelling to do so if one looks for curves of constant amplification rate $c_i =$ const. The required minima in the $(c_r, \alpha)$-plane are even more difficult to locate and Powell's method is absolutely necessary in that case.

In the numerical solution there is another important difficulty. The quantity $\alpha$Re can be large *e.g.* $10^4$ to $10^6$ in the range of interest.

In the case of a constant primary velocity $U(y) = 1$ four particular solutions of the Orr–Sommerfeld equation (2.24a) are

$$\exp(-\alpha y) , \tag{3.5}$$

$$\exp(+\alpha y) , \tag{3.6}$$

$$\exp\{y(\alpha^2 + i\alpha \mathrm{Re}(1-c))^{\frac{1}{2}}\} , \tag{3.7}$$

$$\exp\{-y(\alpha^2 + i\alpha \mathrm{Re}(1-c))^{\frac{1}{2}}\} . \tag{3.8}$$

Now the primary flow along the plane is $y$-dependent, so four particular solutions in that case will not be eq. (3.5) through (3.8) exactly, but one can assume that qualitatively the behaviour will be the same. If we assume that the real part of $(\alpha^2+i\alpha \mathrm{Re}(1-c))^{\frac{1}{2}}$ is positive, the solution (3.8) is a rapidly oscillating function which will dominate the others in the range $y<0$, because $\alpha \mathrm{Re}$ is large.

In eq. (3.4) the particular solutions $\bar{\phi}_1(y)$ and $\bar{\phi}_2(y)$ must themselves be linear combinations of the four particular solutions (3.5) through (3.8). Whatever the initial values (3.2) and (3.3) are, due to rounding errors, truncation *etc.* the functions $\bar{\phi}_1(y)$ and $\bar{\phi}_2(y)$ will eventually behave like the dominant particular solution (3.8) over most of the integration interval. This renders the determination of $A_1$ and $A_2$ in eq. (3.4) very inaccurate or even completely wrong, because the solutions $\bar{\phi}_1(y)$ and $\bar{\phi}_2(y)$ become dependent. This difficulty is dealt with by the method of orthogonalization, devised by Godunov [3] and applied by Wazzan [11] to boundary layer stability. In this method the two solutions $\bar{\phi}_1(y)$ and $\bar{\phi}_2(y)$ are obtained by Runge–Kutta integration over a relatively short interval $[y_k, y_{k+1}]$, subject to certain initial conditions. Then a check is made on the dependency of the two solutions. The dependency is measured by the inner product $\langle\bar{\phi}_1(y), \bar{\phi}_2(y)\rangle_{y=y_{k+1}}$. If necessary the vectors $\bar{\phi}_1$ and $\bar{\phi}_2$ are orthonormalized by the Gram–Schmidt process

$$\bar{\phi}_{1,\mathrm{new}}(y_{k+1}) = \frac{\bar{\phi}_{1,\mathrm{old}}(y_{k+1})}{|\bar{\phi}_{1,\mathrm{old}}(y_{k+1})|}, \tag{3.9}$$

$$\bar{\phi}_{2,\mathrm{new}}(y_{k+1}) = \frac{\bar{\phi}_{2,\mathrm{old}}(y_{k+1}) - \langle\bar{\phi}_{1,\mathrm{new}}(y_{k+1}), \bar{\phi}_{2,\mathrm{old}}(y_{k+1})\rangle\, \bar{\phi}_{1,\mathrm{new}}(y_{k+1})}{|\bar{\phi}_{2,\mathrm{old}}(y_{k+1}) - \langle\bar{\phi}_{1,\mathrm{new}}(y_{k+1}), \bar{\phi}_{2,\mathrm{old}}(y_{k+1})\rangle\, \bar{\phi}_{1,\mathrm{new}}(y_{k+1})|}. \tag{3.10}$$

The new vector solutions (3.9) and (3.10) are orthogonal and normalized. In this way the solutions $\bar{\phi}_1(y)$ and $\bar{\phi}_2(y)$ remain more or less independent, with corresponding accurate determination of $A_1$ and $A_2$ in eq. (3.4). The orthonormalization method was absolutely necessary in the range of interest of the various parameters.

## 4. Results

In fig. 3 the neutral curves taken from Lin's [6, 7] papers are compared with the results obtained by the present author using Lin's incorrect boundary condition in his program. There is a close agreement between the two sets of results for the various angles of inclination. So one can safely assume that both methods of calculation are correct, apart from the treatment of the boundary condition and the consequent different numerical results.

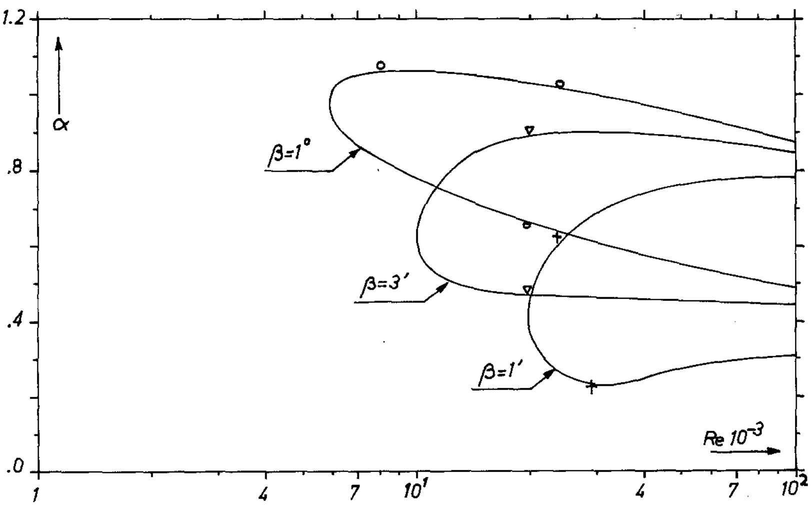


Figure 3. Neutral curves at various $\beta$. Solid lines: S. P. Lin's results. ○, ∇, + present results using Lin's incorrect boundary condition at $\beta=1°$, 3′ and 1′ respectively.

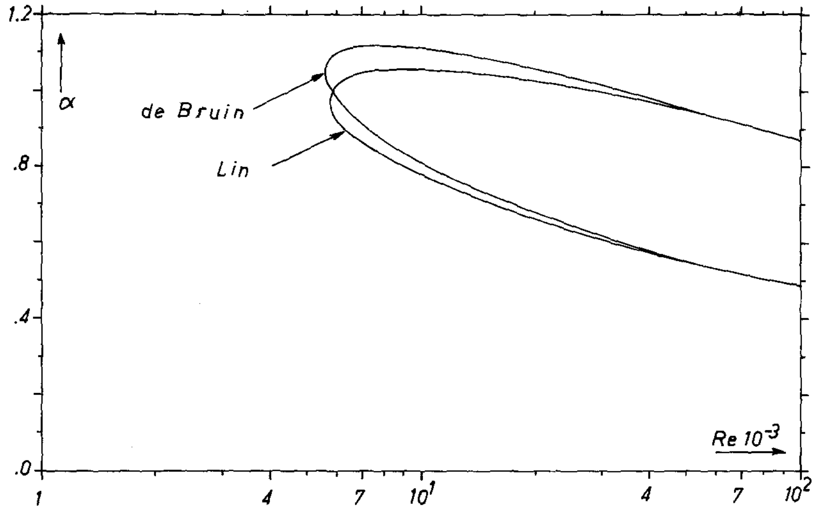


Figure 4. Neutral curve at $\beta = 1°$.

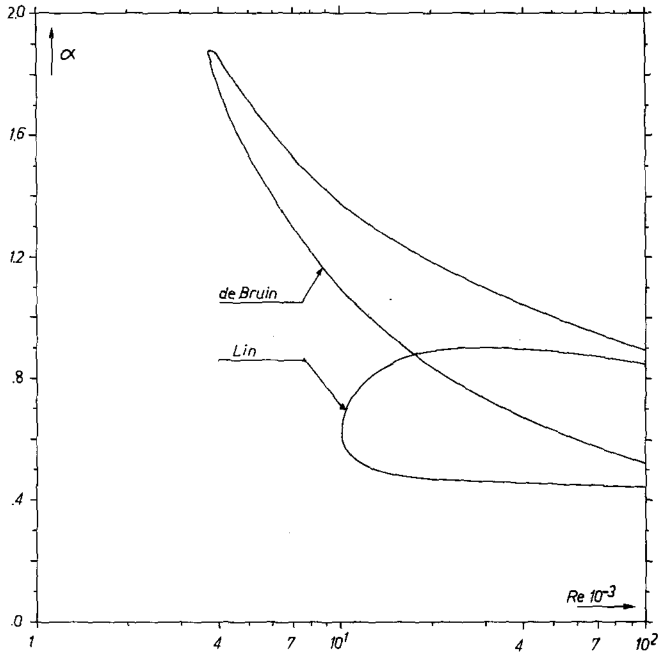


Figure 5. Neutral curve at $\beta = 3'$.

In fig. 4 we show the results for an inclination angle of $\beta = 1°$. The neutral curves are only slightly different either using the correct boundary condition or using Lin's results. The critical Reynolds numbers are almost the same, approximately 5600, but the wavelengths are smaller in the present results. The neutral curve for plane Poiseuille flow as given by Shen [9] would almost coincide with the neutral curve for $\beta = 1°$.

At smaller inclination angles there is a large discrepancy between Lin's results and the present results. In fig. 5 the neutral curve is shown for an inclination angle of $\beta = 3'$. At the lower Reynolds numbers there is complete disagreement with Lin's results. At that inclination angle the critical Reynolds number has been decreased to about 3700.

However, a further decrease in inclination angle does not imply a further decrease in the critical Reynolds number $Re_{cr}$. This can be seen from fig. 6, where the neutral curves are shown for the values of the inclination angle $\beta = 1°, 4', 3', 1', 0.5'$. At extremely small angles the critical

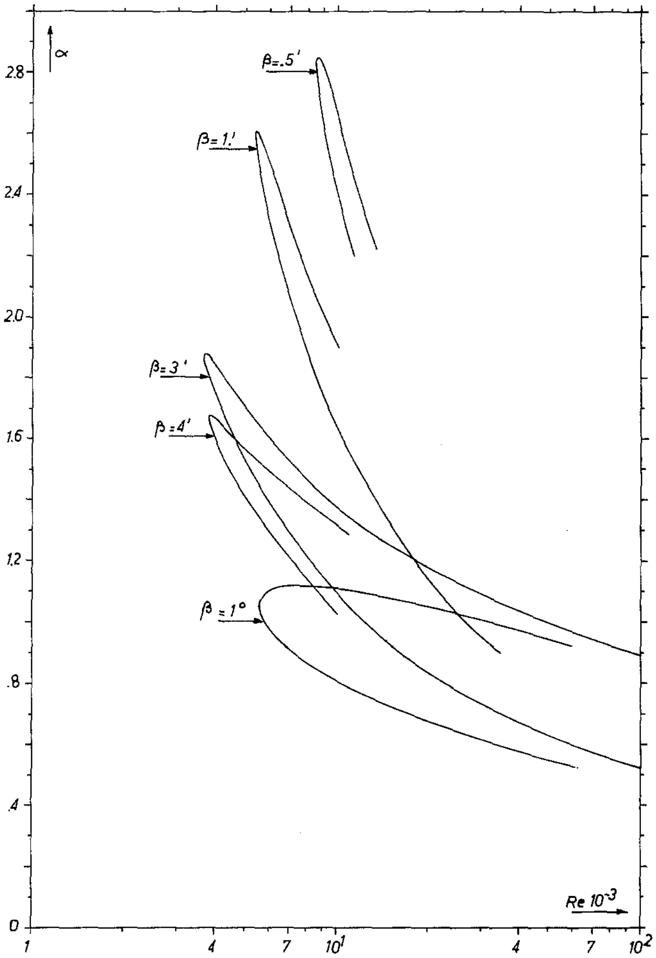


Figure 6. Neutral curves at different $\beta$ using correct boundary conditions.

Reynolds number increases again when decreasing the inclination angle. Whether the critical Reynolds number increased or decreased, the corresponding wave number $\alpha_{cr}$ was increasing when decreasing the inclination angle.

Sometimes it is rather difficult to obtain parts of the neutral curve. Invariably this is the case when starting from a bad guess of the wave velocity $c_r$. If this occurs one has to look for $(c_r, c_i)$ at a fixed set $(\alpha, \text{Re})$, because in that case the rate of convergence is much larger than in the case of looking for $(c_r, \alpha)$ at a fixed $(c_i, \text{Re})$. Using some sets $(\alpha, \text{Re})$ one can get close enough to the neutral curve to use again the tracing program for $(c_r, \alpha)$ at fixed $(\text{Re}, c_i=0)$. Occasionally there is the difficulty that the tracing program jumps from the lower branch to the upper branch or *vice versa*. Imposing smaller steps in the tracing program prevents the occurrence of that difficulty.

In Figures 7 and 8 are shown the eigenfunction and the corresponding Reynolds stress $\overline{uv}$ in the normalized form $2(\phi_i \phi_r' - \phi_r \phi_i')$ for a hard mode and a soft mode. In the case of a soft mode the Reynolds stress does show a continuous and smooth behaviour over the entire depth, because there is no critical layer. In the case of a hard mode however the Reynolds stress shows a rather abrupt change at the critical layer.

Between the critical layer and the free surface the Reynolds stress is vanishing small, with the exception of a region close to the free surface. Over a range of about 5% of the layer thickness there is a sudden rise in Reynolds stress. This behaviour is qualitatively the same at the various inclination angles and Reynolds numbers used in the present calculations, though of course with different numerical values.

Between the free surface and the critical layer the complete solution almost exactly satisfies the inviscid Rayleigh equation. In the wall region and in the critical layer viscous effects become important. Also, the Rayleigh equation is not satisfied near the free surface. Rayleigh's equation applied at the free surface gives

$$\phi''(0) - \alpha^2 - \frac{2}{c-1} = 0 \tag{4.1}$$

whereas the boundary condition on the tangential stress would give

$$\phi''(0) + \alpha^2 - \frac{2}{c-1} = 0\,. \tag{4.2}$$

Apparently the two are contradictory. The boundary condition (4.2) has to be applied to arrive at an eigensolution at all, so the Rayleigh equation can't be satisfied in a small layer near the free surface.

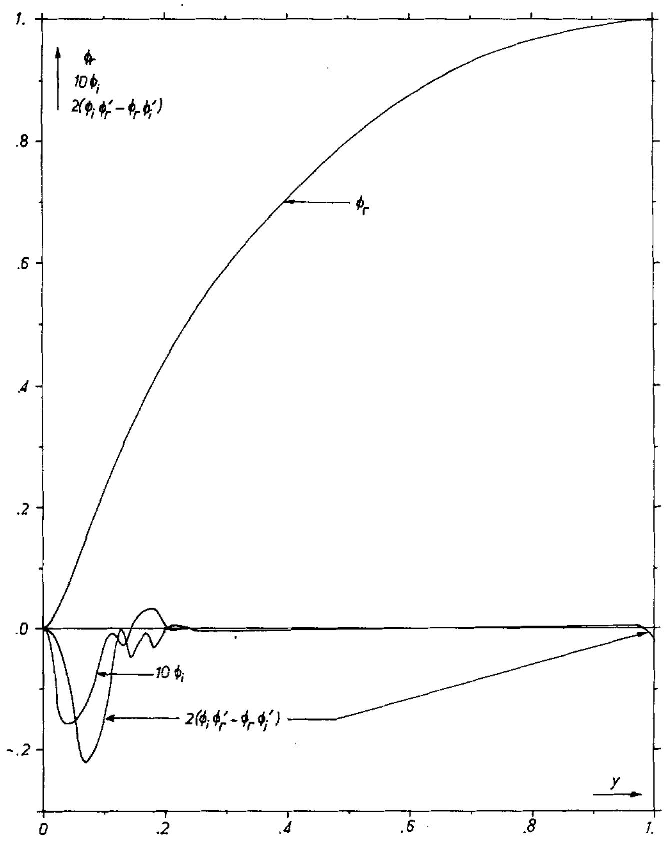


Figure 7. Eigenfunction $\phi$ and the corresponding Reynolds stress Hard mode: $\beta = 1°$, Re = 24065, $\alpha = 1.0354$, $c = 0.2044$.

In figures 9 and 10 the amplification rates $\alpha c_i$ are given for $\beta = 3'$ at Re = 5000 and at Re = 10.000 in the case of a hard mode. In fig. 11 the amplification rate $\alpha c_i$ is given for $\beta = 3'$ and Re = 5000 for the soft mode. The amplification rate for the hard mode is an order of magnitude larger than for the soft mode. The lower branches of the curves of constant $\alpha c_i$ for the soft mode crowd together near the Re-axis. This can be deduced from fig. 11, where the logarithmic distribution on the $\alpha$-axis is necessary for clear view of the amplification rates near the Re-axis.

In table 1 the critical Reynolds numbers are given for the two modes at the various angles of inclination.

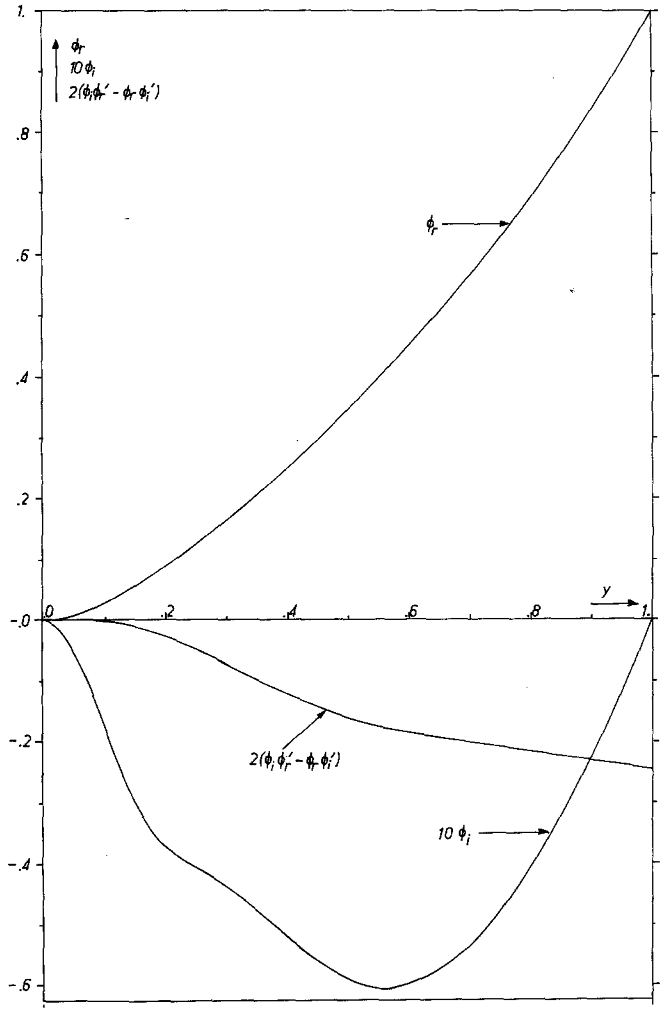


Figure 8. Eigenfunction $\phi$ and the corresponding Reynolds stress Soft mode: $\beta = 1°$, $\mathrm{Re} = 100$, $\alpha = 0.3597$, $c = 1.8141$.

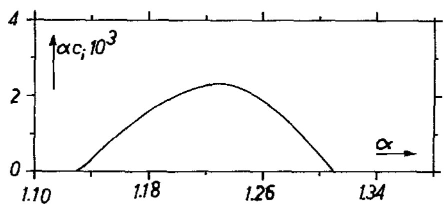


Figure 9. Amplification rate at $\beta = 3'$ and $\mathrm{Re} = 5000$. Hard mode.

## 5. Conclusions

Lin's method and the present method give about the same results when the same incorrect boundary condition is used. So basically Lin's method is correct, however, his numerical results and the conclusions based upon these results are wrong. For the hard mode a decrease in inclination angle $\beta$ gives at first a decrease in the critical Reynolds number, but at very small angles, say $1'$ or less, the critical Reynolds number starts to increase again.

The soft mode has an increasing critical Reynolds number when decreasing $\beta$ according to eq. (1.1). As can be seen from table 1 the critical Reynolds number for the hard mode is greater than that of the soft mode for $\beta > 1'$. However for $\beta = 0.5'$ we have $\mathrm{Re}_{cr,\,hard} < \mathrm{Re}_{cr,\,soft}$. So for

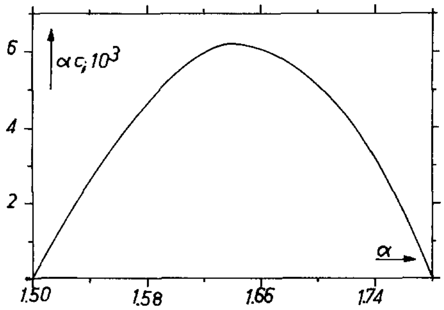


Figure 10. Amplification rate at $\beta=3'$ and $\mathrm{Re}=10^4$. Hard mode.

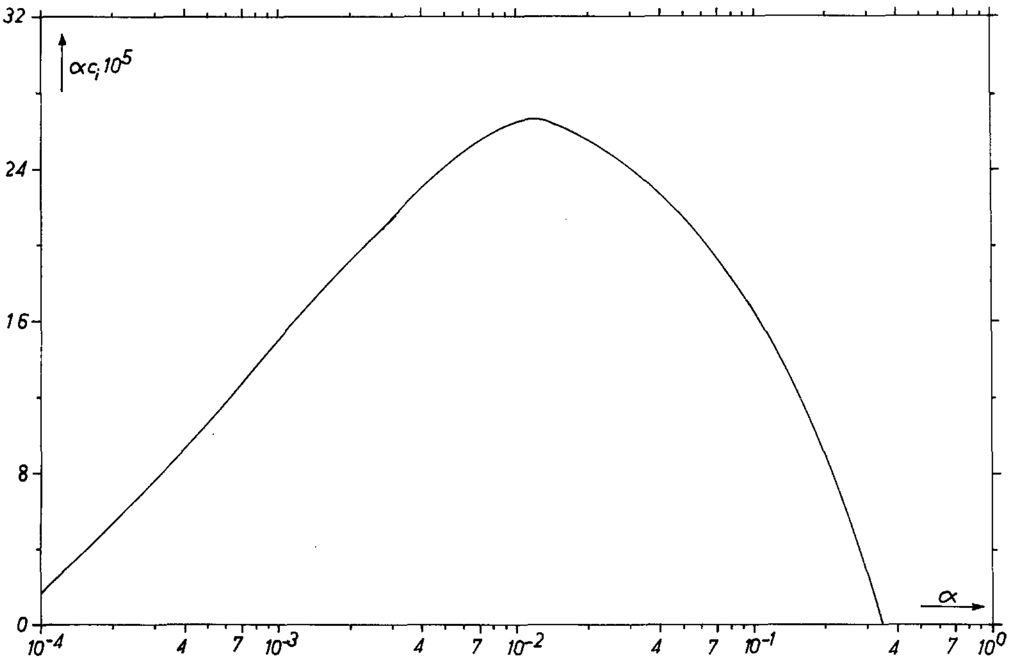


Figure 11. Amplification rate of the soft mode. $\beta=3'$, $\mathrm{Re}=5000$.

TABLE 1

*Critical Reynolds number as a function of* $\beta$

| $\beta$ | $Re_{crit}$ | |
|---|---|---|
| | *soft* | *hard* |
| 1° | 72 | 5600 |
| 4′ | 1070 | 3900 |
| 3′ | 1430 | 3700 |
| 1′ | 4300 | 5400 |
| 0.5′ | 8600 | 8500 |

very small angles, say less than 0.5′, the stability is probably governed by the hard mode. For angles of inclination $\beta > 1°$ the stability is governed by the soft mode.

In the range $1' < \beta < 1°$ the soft mode is unstable at smaller Reynolds number than the hard mode. However, the amplification rates for the soft modes are an order of magnitude smaller than the amplification rates for the hard modes, as can been seen from figures 9 and 11.

So it is conceivable that in the actual experiment the amplification of the soft mode remains relatively unimportant over the amplification rate of the hard mode. In that $\beta$-range one must be careful upon comparing theoretical and experimental results. Also, for that comparison, it would have been better to solve the spatial stability problem, in which there is decay or amplification in space rather than in time. However, often the wave velocities do not differ much for the two stability problems, at least in the neighbourhood of the neutral curve.

What becomes apparent from the behaviour of the Reynolds stress is that for the hard mode there are three layers where viscosity must be taken into account:

(i) a layer near the wall, to satisfy the no-slip condition
(ii) a critical layer where $U=c$
(iii) a layer near the free surface, where the Rayleigh equation is not a valid approximation to the Orr–Sommerfeld equation.

Experimental results for $\beta$ less than 0.5° could not be found in the literature, so a comparison of theoretical results and the experiment cannot be given at this stage.

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